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NOTE

On the Number of Blocks in a Generalized Steiner System

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derived. For $t=2$, this inequality is the well known De Bruijn–Erdős inequality. For $t>2$ it has the same order of magnitude as the Wilson–Petrenjuk inequality for Steiner systems with constant block size. The point of this note is that the inequality is very easy to derive and does not seem to be known. A stronger inequality was derived in 1969 by Woodall (*J. London Math. Soc. (2)* **1**, 509–519), but it requires Lagrange multipliers in the proof.

We consider a so-called *generalized Steiner system* $t-(n, *, 1)$, i.e., a collection $\mathscr{B}$ of subsets (*blocks*) of an n-set $\mathscr{P}$ (of n *points*) with the property that *every* t-subset of $\mathscr{P}$ is contained in *exactly one* block in $\mathscr{B}$.

We represent such a system by a (0,1)-matrix A of size b by n, with $b := |\mathscr{B}|$, where the ith row of A is the characteristic function of the ith block $B_i \in \mathscr{B}$.

A generalized Steiner system is called *trivial* if $|\mathscr{B}| = 1$.

Definition. We define $\beta_{t,n}$ to be the *minimal* number of blocks in a nontrivial system $t-(n, *, 1)$.

Theorem. *For $t \geqslant 2$ we have*

$$\beta_{t,n}(\beta_{t,n}-1) \geqslant t\binom{n}{t}.$$

Proof. The proof is by induction. The case $t=2$ is the well known Erdős-De Bruijn inequality (if $t=2$ and $|\mathscr{B}|>1$, then $|\mathscr{B}| \geqslant n$; cf. [2,

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Theorem 19.1]). Now, assume that the theorem has been proved for $t-1$ and all n. Consider the matrix A of a $t-(n, *, 1)$. If any column of A has only one 1, then the system is trivial. So, for any point, the derived design with respect to this point is a non-trivial $(t-1)-(n-1, *, 1)$ system. This implies that all columns of A have at least $\beta_{t-1, n-1}$ ones. We now count *pairs* $(a_{i,k}, a_{j,k})$ equal to $(1, 1)$ with $1 \leqslant i < j \leqslant b$. By first choosing the pair (i, j), we find at most $t-1$ such pairs $(1, 1)$. So, the total number is at most

$$(t-1)\binom{b}{2}.$$

In any column of A, we find at least

$$\binom{\beta_{t-1, n-1}}{2}$$

such pairs. It follows that

$$(t-1)\binom{\beta_{t,n}}{2} \geqslant n\binom{\beta_{t-1,n-1}}{2} \geqslant \frac{n(t-1)}{2}\binom{n-1}{t-1},$$

i.e.,

$$\beta_{t,n}(\beta_{t,n}-1) \geqslant t\binom{n}{t}. \quad \blacksquare$$

Remark. Note that the t-subsets of a $(t+1)$-set form a $t-(t+1, *, 1)$-system for which equality holds in the theorem.

In general this bound is weak. However, the result is easy to derive and certainly deserves to be an exercise in combinatorics books. If we fix t, the inequality is a diophantine equation in β and n which probably has very few solutions. So equality is not to be expected except for the case already mentioned. For $t=3$ and $n=5$ we find $\beta(\beta-1) \geqslant 30$, where $\beta=6$ would give equality. However, it is easy to see that a $3-(5, *, 1)$ design with 6 blocks does not exist. For $t=3$, we do find the interesting fact that β grows like $n^{3/2}$. If $t=3$ and $n=8$, we find that $\beta > 13.47$, so a design with these parameters must have at least 14 blocks. Indeed, there is a $S(3, 4, 8)$ with 14 blocks.

We now compare the result with the well known Wilson–Petrenjuk inequality ($\beta \geqslant \binom{n}{s}$ if $t=2s$; cf. [2, Theorem 19.8]). Clearly, Wilson–Petrenjuk is stronger. For example, for $t=4$ it yields as the right-hand side in our inequality $6\binom{n+1}{4}$ instead of $4\binom{n}{4}$. Note, however, that in both bounds the rate of growth of the bound for β (for fixed t) is as $n^{t/2}$.

We also compare our bound with a result due to Woodall (cf. [3]). This states that

$$\beta \geqslant \binom{n}{t} \Big/ \binom{k}{t},$$

where k is the larger root of

$$(k-t+2)(k-t+1)=n-t+1$$

and the binomial coefficient is interpreted in the usual way if k is not an integer. For $t=2$, this is again the De Bruijn–Erdős inequality. This bound is more difficult to derive but it is stronger than our simple inequality. In our example ($t=3$, $n=8$) it yields $\beta \geqslant 14$ which is exact. For $t=3$ and $n=5$ it yields $\beta > 6.05$ showing the nonexistence of the $3-(5, *, 1)$ mentioned above.

We remark that for Steiner systems with $t=4$ and $n>23$, Woodall's bound is larger than the Wilson–Petrenjuk bound, showing that for these parameters a tight design cannot exist.

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